Int. J. Heat Mass Transfer. Vol. 16, pp. 195–211. Pergamon Press 1973. Printed in Great Britain

HEAT TRANSFER WITH MELTING OR FREEZING IN A WEDGE

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(*Received* 11 *November* 1971 *and in revised form* 2 *May* 1972)

Abstract—The objective of this research was to obtain an analytical solution to predict the temperature distribution and the motion of the interface in a pure liquid or eutectic alloy solidifying or melting in a wedge shaped enclosure. In the analysis it was assumed that the initial temperature of the medium is uniform and the surfaces of the wedge are maintained at uniform, but not necessarily equal, temperatures.

The solution to the problem was obtained by the superposition of the solutions to two auxiliary problems. The first of these was the problem of heat conduction without phase change, but with the same initial and boundary conditions as those of the actual problem. The second subproblem was that of heat conduction with change of phase, but with the initial and boundary temperatures equal to zero. In the later subproblem the latent heat liberated due to the phase change was represented by a moving surface source along the interface. The temperature distributions for these auxiliary problems were obtained by using Green's function.

The results of the analytical solution presented here agree with previously published experimental and numerical results for special cases to within 5 per cent.

NOMENCLATURE

a,	characteristic parameter in the equation of the hyperbola;
b,	arbitrary constant;
c,	specific heat;
$\vec{i}_r, \vec{i}_\phi$	unit normals in r and ϕ directions;
k,	thermal conductivity;
L,	latent heat of fusion;
n,	integer constant;
P,	temperature distribution due to conduction only;
Q,	temperature distribution due to moving surface source;
r,	radial co-ordinate;
r',	integration variable;
r^*,	dimensionless radial co-ordinate;
t,	time;
t',	integration time variable;
T,	dimensionless temperature field;
T_F,	fusion temperature;
T_i,	initial temperature;
T_i^*,	dimensionless initial temperature;
T_L,	temperature in the liquid region;
T_L^*,	dimensionless temperature in the liquid region;
T_s,	temperature in the solid region;
T_s^*,	dimensionless temperature in the solid region;
T_{wx}, T_{wy},	surface temperatures at $\phi = 0$ and $\phi = \phi_0$, respectively;
α,	thermal diffusivity;
β,	latent heat to sensible heat ratio $(L/\rho c(T_F - T_{wx}))$;
ν, η, ξ, τ,	integration variables;
$\lambda, \lambda_x, \lambda_y, \lambda', \lambda'_x, \lambda'_y$,	characteristic distances for one-dimensional interfaces;
ϕ,	angular co-ordinate.

Subscripts

F,	fusion quantity;
i,	initial quantity;
L,	liquid quantity;
p,	index;

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s,	solid quantity.

Functions

erf x,	error function;
erfc x,	complementary error function;
$\exp(x)$ or e^x,	exponential function;
$I_p(x)$,	modified Bessel function of first kind;
$J_p(x)$,	Bessel function of first kind;
$G(r,\phi,r',\phi';t,t')$,	Green's function;
$R(\phi,t), R'(\phi)$,	functions for the interfacial curve.

1. INTRODUCTION

HEAT transfer with a change of state due to melting or freezing occurs in many engineering systems, e.g. solidification of castings, freezing and thawing of foodstuffs and soil, constant-temperature sinks for energy-storage devices, and geophysical phenomena of ice formation. In all these problems there exists a moving free boundary, separating the liquid and the solid phases, whose location and shape are not known *a priori*. Since the temperature distributions in the two phases depends on the location and shape of this interface, it is necessary to consider the temperature distributions and the interface motion simultaneously in any analyses of such problems. Moreover, these systems are non-linear in nature due to the energy balance at the moving interface [1].

Two excellent reviews [2, 3] of the field show that till 1963 published work in heat transfer with melting or freezing was limited to problems which can be described in terms of a single space variable. In most practical situations, however, one encounters two- or three-dimensional problems and since 1963 several investigations dealing with two- or three-dimensional systems have been reported in the literature [4–10].

Allen and Severn [4] and Poots [5] presented approximate solutions for the solidification of a liquid initially at freezing temperature in a square prism with the boundaries of the prism maintained at a constant temperature. Springer and Olson [6] developed a finite difference scheme for axisymmetric solidification or melting of materials contained within two concentric cylinders of finite length. Their difference scheme has provision for variable thermal properties and various boundary conditions. Sikarskie and Boley [7] obtained solutions for two problems in which two-dimensional effects were introduced by spatial variations of heating or cooling conditions along one boundary surface of a slab. Rathjen [8] has presented an analytical solution for melting or freezing in an infinite rectangular internal corner with the material initially at a uniform temperature and the surfaces maintained at a constant temperature, and compared his analytical results with those of a numerical scheme for the same problem. Jiji *et al.* [9] published some experimental results for freezing of water in an internal corner and compared their experimental results with those obtained numerically by Rathjen [8]. Recently, Lazaridis [10] published a numerical scheme to treat multidimensional problems with boundary conditions of constant temperature and Newtonian cooling.

This paper presents analytical solutions for melting or freezing of materials in wedge shaped enclosures with wedge angle between 0- and 360-degrees. The postulates used in the solutions are: (1) The material has a sharp fusion temperature. (2) The thermal and physical properties of the solid and liquid phases are constant and are independent of temperature. (3) The initial temperature of the material inside the wedge is uniform and the surfaces of the wedge are kept at constant, but not necessarily equal, temperatures. The analytical solutions presented in this paper can be used directly in many engineering applications and can also serve as "starting" solutions for numerical analyses.

2. METHOD OF ANALYSIS

Suppose a liquid at a uniform temperature T_i greater than or equal to its freezing temperature T_F, fills a wedge shaped space as shown in Fig. 1. At time zero ($t = 0$), the two faces of the wedge ($\phi = 0$ and $\phi = \phi_0$) are suddenly brought to constant temperatures lower than T_F. Solidification starts at both faces and as time progresses the boundary between the solid and liquid phases moves into the liquid phase. If the density of each phase is uniform, heat is transferred in both phases only by conduction and the problem can be stated mathematically as follows:

$$\frac{\partial^2 T_s}{\partial r^2} + \frac{1}{r}\frac{\partial T_s}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T_s}{\partial \phi^2} = \frac{1}{\alpha_s}\frac{\partial T_s}{\partial t} \quad : 0 < r \leqslant R(\phi, t), 0 < \phi < \phi_0 \qquad (1)$$

$$\frac{\partial^2 T_L}{\partial r^2} + \frac{1}{r}\frac{\partial T_L}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T_L}{\partial \phi^2} = \frac{1}{\alpha_L}\frac{\partial T_L}{\partial t} \quad : R(\phi, t) \leqslant r < \infty, 0 < \phi < \phi_0 \qquad (2)$$

$$T_s = T_{wx} \qquad \phi = 0, r > 0; t > 0 \qquad (3)$$

$$T_s = T_{wy} \qquad \phi = \phi_0, r > 0; t > 0 \qquad (4)$$

$$T_L = T_i \qquad 0 < \phi < \phi_0, \quad 0 < r < \infty \quad \text{at } t = 0 \qquad (5)$$

$$T_L = T_i \qquad 0 < \phi < \phi_0, \quad r \to \infty; t > 0 \qquad (6)$$

$$T_s = T_L = T_F \qquad 0 < \phi < \phi_0, \quad r = R(\phi, t); t > 0 \qquad (7)$$

$$\left(k_s \frac{\partial T_s}{\partial r} - k_L \frac{\partial T_L}{\partial r}\right)\left[1 + \frac{1}{R^2}\left(\frac{\partial R}{\partial \phi}\right)^2\right] = \rho L \frac{\partial R}{\partial t} \qquad (8)$$

$$\text{at } r = R(\phi, t), 0 < \phi < \phi_0; t > 0$$

$$R = 0 \qquad 0 < \phi < \phi_0, t = 0 \qquad (9)$$

where $T_s(r, \phi, t)$ and $T_L(r, \phi, t)$ are the temperature distributions in the solid and the liquid phases respectively, $R(\phi, t)$ is the function which specifies the shape and location of the unknown phase boundary, T_{wx} and T_{wy} are the surface temperatures, α_s and α_L are thermal diffusivities in the liquid and solid phases respectively, and L is the latent heat of fusion.

Equations (7)–(9) describe the conditions at the interface. Equation (7) states that the temperature at the interface between the solid and the liquid phase equals the freezing temperature. Equation (8) represents the energy balance at the interface, i.e. the heat conducted away through the solid phase equals the heat reaching the interface from the liquid phase plus the latent heat generated due to solidification. Equation (9) gives the initial position of the interface.

With subscripts s and L interchanged, Equations (1)–(9) represent the mathematical statement of a solid material melting in a wedge shaped enclosure. For the melting problem the surface temperatures T_{wx} and T_{wy} are higher than the fusion temperature and the latent heat in equation (8) is negative because heat is absorbed at the interface during melting.

Similarity in the problem

As shown below, a similarity transformation can reduce the number of independent variables for the problem from three (r, ϕ, t) to two $[r/\sqrt{(4\alpha t)}]$ and ϕ. As a result of the similarity in the system the equation of the interface can be written as

$$r = R(\phi, t) = \sqrt{(4\alpha t)}R'(\phi) \qquad (10)$$

where R' is a function of ϕ only.

To show that similarity exists, define new function $\tilde{T}_s$, $\tilde{T}_L$, $\tilde{R}$ (for $b > 0$) as follows:

$$\tilde{T}_s(r, \phi, t; b) = T_s(br, \phi, b^2 t)$$

$$\tilde{T}_L(r, \phi, t; b) = T_L(br, \phi, b^2 t)$$

$$\tilde{R}(r, \phi, t; b) = \frac{1}{b} R(\phi, b^2 t).$$

It can easily be verified that these functions also satisfy equations (1)–(9). The new functions $\tilde{T}_s$, $\tilde{T}_L$ and $\tilde{R}$ must, therefore, be equal to T_s, T_L and R, respectively, because both satisfy the same initial-boundary value problem:

$$\tilde{T}_s(r, \phi, t; b) \equiv T_s(br, \phi, b^2 t) = T_s(r, \phi, t)$$

$$\tilde{T}_L(r, \phi, t; b) \equiv T_L(br, \phi, b^2 t) = T_L(r, \phi, t)$$

$$\tilde{R}(\phi, t; b) \equiv \frac{1}{b} R(\phi, b^2 t) = R(\phi, t).$$

These three equations are identities in r, ϕ, t and b. Thus, they hold in particular for $b^2 t = 1/4\alpha$, which gives:

$$T_s(r, \phi, t) = T_s[r/\sqrt{(4\alpha t)}, \phi, 1/4\alpha] = T'_s[r/\sqrt{(4\alpha t)}, \phi]$$

$$T_L(r, \phi, t) = T_L[r/\sqrt{(4\alpha t)}, \phi, 1/4\alpha] = T'_L[r/\sqrt{(4\alpha t)}, \phi]$$

$$R(\phi, t) = \sqrt{(4\alpha t)} R(\phi, 1/\alpha) = \sqrt{(4\alpha t)} R'(\phi).$$

T'_s, T^*_L and R', which constitute the solution to the problem, are therefore only functions of the two independent variables $r/\sqrt{(4\alpha t)}$ and ϕ.

In the method of solution presented below, the similarity condition has not been used at the outset to reduce the number of independent variables in the diffusion equations. It will be shown later, however, that the independent variables r, ϕ, t in the solution can be combined into the two variables $r/\sqrt{(4\alpha t)}$ and ϕ. The reason for presenting the similarity argument at this point is to show that the equation of the interface is given by equation (10). This equation implies that the moving interface can be plotted into a stationary one by using $r/\sqrt{(4\alpha t)}$ and ϕ as the coordinates.

Method of solution

Introducing the dimensionless temperature variables

$$T^*_s = \frac{T_s - T_F}{T_F - T_{wx}} \tag{11a}$$

$$T^*_L = \frac{k_L}{k_s} \frac{T_L - T_F}{T_F - T_{wx}} \tag{11b}$$

equations (1)–(9) become:

$$\frac{\partial^2 T^*_s}{\partial r^2} + \frac{1}{r}\frac{\partial T^*_s}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T^*_s}{\partial \phi^2} = \frac{1}{\alpha_s}\frac{\partial T^*_s}{\partial t} \tag{12}$$

$$\frac{\partial^2 T^*_L}{\partial r^2} + \frac{1}{r}\frac{\partial T^*_L}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T_L}{\partial \phi^2} = \frac{1}{\alpha_L}\frac{\partial T^*_L}{\partial t} \tag{13}$$

$$T^*_s = T^*_{wx} = -1 \quad \phi = 0, r > 0; t > 0 \tag{14}$$

$$T^*_s = T^*_{wy} \quad \phi = \phi_0, r > 0; t > 0 \tag{15}$$

$$T^*_L = T^*_i = \frac{k_L}{k_s}\frac{T_i - T_F}{T_F - T_{wx}} \quad 0 < \phi < \phi_0, r > 0; t = 0 \tag{16}$$

$$T^*_L = T^*_i \quad 0 < \phi < \phi_0, r \to \infty; t > 0 \tag{17}$$

$$T^*_s = T^*_L = 0 \quad 0 < \phi < \phi_0, r = R(\phi, t); t > 0 \tag{18}$$

$$k_s\left(\frac{\partial T^*_s}{\partial r} - \frac{\partial T^*_L}{\partial r}\right)\left(1 + \frac{1}{R}\left(\frac{\partial R}{\partial \phi}\right)^2\right) = \frac{\rho L}{(T_F - T_{wx})}\frac{\partial R}{\partial t} \tag{19}$$

$$\text{at } r = R(\phi, t), 0 < \phi < \phi_0; t > 0$$

$$R = 0 \quad 0 < \phi < \phi_0; t = 0. \tag{20}$$

To make the problem analytically tractable, assume that the thermal diffusivities in the liquid and solid phases are equal, i.e. $\alpha_s = \alpha_L = \alpha$. Under this assumption, the heat conduction process in the solid and liquid phases described by equations (12) and (13) can be represented by the single equation:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial T}{\partial \phi^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{21}$$

with $T = T^*_s$ in the solid phase and
with $T = T^*_L$ in the liquid phase.

The assumption that the thermal diffusivities are equal is not necessary when the liquid is initially at the freezing temperature ($T^*_i = 0$) because there is no heat diffusion in the liquid phase for this case. An empirical method to correct for the error introduced by this assumption when the liquid is initially a temperature above the freezing point will be presented later.

The problem now reduces to solving equation (21). The various conditions to be satisfied by $T(r, \phi, t)$ are given by equations (14)–(20) in which T is substituted for T^*_s and T^*_L. The method of superposition of the solutions of two separate problems will be used to solve the problem stated above. The two subproblems used to construct the solutions are:

(a) The problem of heat conduction, without change of phase, in a medium initially at T_i^* with the wedge surfaces $\phi = 0$ and $\phi = \phi_0$ changed to, and maintained at, constant temperatures T_{wx}^* and T_{wy}^*, respectively, for all times $t > 0$.

(b) The problem of a moving source of heat at the interface $[r = R(\phi, t)]$ in a medium initially at zero temperature with the wedge surfaces maintained at zero temperatures. The moving source at the interface replaces the latent heat generated due to the phase change.

The superposition solution is stated mathematically below:

$$T(r, \phi, t) = P(r, \phi, t) + Q(r, \phi, t) \quad (22)$$

where $P(r, \phi, t)$ is the solution to the conduction problem without change of phase and $Q(r, \phi, t)$ is solution to the "moving source" problem.

The pure conduction problem can be posed as

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \phi^2} = \frac{1}{\alpha}\frac{\partial P}{\partial t} \quad (23)$$

with the conditions

$$P(r, 0, t) = T_{wx}^* = -1 \quad (24)$$

$$P(r, \phi_0, t) = T_{wy}^* \quad (25)$$

$$P(r, \phi, 0) = T_i^* \quad (26)$$

while the function $Q(r, \phi, t)$ must satisfy the equation

$$\frac{\partial^2 Q}{\partial r^2} + \frac{1}{r}\frac{\partial Q}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Q}{\partial \phi^2} = \frac{1}{\alpha}\frac{\partial Q}{\partial t} \quad (27)$$

with the conditions

$$Q(r, 0, t) = 0 \quad (28)$$

$$Q(r, \phi_0, t) = 0 \quad (29)$$

$$Q(r, \phi, 0) = 0. \quad (30)$$

P-solution

The P-solution was obtained by means of Green's function. The appropriate Green's function $G(r, \phi, r', \phi'; t, t')$ must satisfy the conduction equation with a homogeneous boundary condition of the first kind subject to zero initial conditions. Carslaw and Jaeger [1] give the following expression for the Green's function in a wedge shaped space:

$$G = \frac{2}{\phi_0}\sum_p \sin p\phi \sin p\phi' \times \int_0^\infty \xi \exp[\alpha\xi^2(t - t')]\, J_p(\xi v).J_p(\xi r')\, d\xi. \quad (31a)$$

An equivalent expression for equation (31a) in somewhat more convenient form is:

$$G \equiv \frac{1}{Q_0\alpha(t - t')}\sum_p \sin p\phi \sin p\phi' \times \exp\left(-\frac{r^2 + r'^2}{4\alpha(t - t')}\right).I_p\left(\frac{rr'}{2\alpha(t - t')}\right) \quad (31b)$$

where $p = n\pi/\phi_0$, $n = 1, 2, 3, \ldots \infty$,

I_p = Modified Bessel function of first kind of order p and J_p = Bessel function of first kind of order p.

According to Ozisik [11], the expression for $P(r, \phi, t)$ can be written in terms of $G(r, \phi, r', \phi'; t, t')$ as follows:

$$P(r, \phi, t) = \int_{r'=0}^{\infty}\int_{\phi=0}^{\phi_0} G|_{t'=0}\, T_i^*.r'.d\phi'\, dr' + \alpha\int_{t'=0}^{t} dt' \int_0^\infty \left(\frac{1}{r'}\frac{\partial G}{\partial \phi'}\right)\Bigg|_{\phi'=0} T_{wx}^*\, dr' - \alpha\int_{t'=0}^{t} dt' \int_0^\infty \left(\frac{1}{r'}\frac{\partial G}{\partial \phi'}\right)\Bigg|_{\phi'=\phi_0} .T_{wy}^*\, dr'. \quad (32)$$

The first term on the right-hand side of equation (32) accounts for the effect of the initial temperature whereas the second and the third terms together represent the effects of the boundary conditions.

The first term in equation (32) is evaluated by substituting for G from equation (31b). With this substitution the first term becomes

$$\frac{T_i^*}{\phi_0 \alpha t} \int_{r'=0}^{\infty} \int_{\phi'=0}^{\phi_0} \sum_p \sin p\phi \sin p\phi' \times \exp\left(-\frac{r^2 + r'^2}{4\alpha t}\right) . I_p\left(\frac{rr'}{2\alpha t}\right) . r' \, d\phi' . dr'.$$

This expression can be reduced further by integrating with respect to ϕ' and then substituting the dimensionless variables $v = r'/\sqrt{(4\alpha t)}$ and $r^* = r/\sqrt{(4\alpha t)}$; this yields the expression

$$\frac{4T_i^*}{\phi_0} \sum_p \frac{\sin p\phi}{p} [1 - \cos p\phi_0] \times \int_0^{\infty} v \exp\left[-(r^{*2} + v^2)\right] . I_p(2r^* v) \, dv. \quad (33)$$

The integral in the preceding expression has to be evaluated numerically. For convenience in computation the order of summation and integration will be interchanged. Making this change and reordering some of the terms, the expression can be written as:

$$\frac{8T_i^*}{\phi_0} \int_0^{\infty} v \, e^{-(r^* - v)^2} \times \left[\sum_{p'} \frac{\sin p'\phi}{p'} e^{-2vr^*} . I_{p'}(2vr^*)\right] . dv$$

where $p' = (2n + 1)\pi/\phi_0$ and $n = 0, 1, 2, \ldots \infty$.

The integral was evaluated numerically by using the forty-point Gaussian–Legendre quadrature formula [14]. For this purpose the upper limit of the integral was replaced by a finite limit either equal to $(r^* + 5)$ or two times r^*, whichever was greater. The infinite series in the brackets converges for finite values of v and r^* because the function $e^{-2vr^*} I'_p(2vr^*)$ decreases monotonically with increasing p'. The infinite series was truncated at that value of p' for which $e^{-2vr^*} I'_p(2vr^*) < 10^{-6}$.

The difference between the second and third terms on the right-hand side of equation (32), which accounts for the effects of the boundary conditions, was evaluated by substituting equation (31a) for G. This substitution yields

$$\frac{2\alpha}{\phi_0} \sum_p (T_{wx}^* - T_{wy}^* \cos p\phi_p) . p . \sin p\phi \times \int_0^t dt' \int_0^{\infty} \frac{dr'}{r'} \int_0^{\infty} \xi \exp\left[-\alpha\xi^2(t - t'\right] \times J_p(\xi r) \, J_p(\xi r') . d\xi.$$

After integration with respect to t' and r' the above expression reduces to

$$\frac{2}{\phi_0} \sum_p (T_{wx}^* - T_{wy}^* \cos p\phi_0) . \sin p\phi \times \left[\int_0^{\infty} \frac{J_p(\xi r)}{\xi} d\xi - \int_0^{\infty} \frac{e^{-\alpha\xi^2 t}}{\xi} J_p(\xi r) \, d\xi\right].$$

If the first integral in the brackets is evaluated as shown in [12], the preceding expression becomes

$$\frac{2}{\phi_0} \sum_p T_{wx}^* \frac{\sin p\phi}{p} - \frac{2}{\phi_0} \sum_p T_{wy}^* \cos p\phi_0 \frac{\sin p\phi}{p} - \frac{2}{\phi_0} \sum_p (T_{wx}^* - T_{wy}^* \cos p\phi_0) \sin p\phi \int_0^{\infty} \frac{J_p(\xi r)}{\xi} e^{-\alpha\xi^2 t} \, d\xi.$$

Some of the infinite series in the above expression can be summed by using the relations [13]

$$\sum_{n=1}^{\infty} \frac{\sin n\pi(\phi/\phi_0)}{h} = \pi\left(\frac{1}{2} - \frac{\phi}{2\phi_0}\right) \qquad 0 < \phi/\phi_0 < 2$$

and

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\pi(\phi/\phi_0)}{n}=\frac{\pi}{2}\frac{\phi}{\phi_0}$$

$$-1<\phi/\phi_0<1.$$

Substituting these series and letting $u = \xi r$, the original expression can be written as

$$T^*_{wx}\left(1-\frac{\phi}{\phi_0}\right)+T^*_{wy}\left(\frac{\phi}{\phi_0}\right)$$
$$-\frac{2}{\phi_0}\sum_p(T^*_{wx}-T^*_{wy}\cos p\phi_0)$$
$$\times\sin p\phi\,.\int_0^{\infty}\frac{J_p(u)}{u}\mathrm{e}^{-(\alpha t/r^2)u^2}\mathrm{d}u\, r>0.$$

The integrals in the above expression can be evaluated by using the relation [12]

$$\int_0^{\infty}\frac{J_p(u)}{u}\mathrm{e}^{-(\alpha t/r^2)u^2}\,\mathrm{d}u$$
$$=\frac{\sqrt{\pi}}{4p}.\frac{r}{\sqrt{\alpha t}}\mathrm{e}^{-r^2/8\alpha t}\left[I_{(p+1)/2}\left(\frac{r^2}{8\alpha t}\right)+T_{(p-1)/2}\left(\frac{r^2}{8\alpha t}\right)\right].$$

Finally, the difference between the second and third terms on the right-hand side of equation (32) is given in terms of dimensionless space variable r^*

$$T^*_{wx}\left(1-\frac{\phi}{\phi_0}\right)+T^*_{wy}\frac{\phi}{\phi_0}$$
$$-\frac{\sqrt{\pi}}{\phi_0}r^*\sum_p(T^*_{wx}-T^*_{wy}\cos p\phi_0)\,.\frac{\sin p\phi}{p}$$
$$\times\mathrm{e}^{-r^{*2}/2}\left[I_{(p+1)/2}\left(\frac{r^{*2}}{2}\right)+I_{(p-1)/2}\left(\frac{r^{*2}}{2}\right)\right].\quad(34)$$

The infinite series in the third term of equation (34) is truncated at that value of p for which

$$\mathrm{e}^{-(r^{*2}/2)}\left[I_{(p-1)/2}(r^{*2}/2)+I_{(p+1)/2}(r^{*2}/2)\right]<10^{-6}.$$

Combining (33) and (34), the P-solution can be written as

$$P(r^*,\phi)=\frac{8T_i^*}{\phi_0}\int_0^{\infty}v\mathrm{e}^{-(r^*-v)^2}$$
$$\times\left[\sum_{p'}\frac{\sin p'\phi}{p'}\mathrm{e}^{-2\bar{v}r^*}I_{p'}(2\bar{v}r^*)\right]\mathrm{d}v$$
$$+T^*_{wx}\left(1-\frac{\phi}{\phi_0}\right)+T^*_{wy}\left(\frac{\phi}{\phi_0}\right)$$
$$-\frac{\sqrt{\pi}}{\phi_0}r^*\sum_p(T^*_{wx}-T^*_{wy}\cos p\phi_0).\frac{\sin p\phi}{p}$$
$$\times\mathrm{e}^{-r^{*2}/2}\left[T_{(p+1)/2}(r^{*2}/2)+T_{(p-1)/2}\left(\frac{r^{*2}}{2}\right)\right].$$

(35)

As a check, the P-solution computed by the above method was compared with closed form solutions available for 180- and 90-degree corners [1] and it was found that the results agreed within five decimal places.

Q-solution

The Q-solution can be obtained by considering a "moving surface source" of heat in the region $r > 0$, $0 < \phi < \phi_0$ along the interface $r = R(\phi, t)$. This heat source has physically the same effect on the temperature field as the phase change and replaces the latent-heat terms in the original problem statement for the system. The temperature distribution due to such a moving-surface source can be obtained by using Green's function for the problem.

It is known that the Green's function $G(r, \phi, r', \phi'; t, t')$ gives the temperature distribution due to an instantaneous line source of *unit* strength at (r', ϕ') liberating (or absorbing) heat at $t = t'$ in a medium initially at zero temperature with the wedge surfaces maintained at zero temperatures. Thus, the temperature distribution due to the heat liberated by the moving surface-source along the interface in time interval $\mathrm{d}t'$(at $t = t'$) can be obtained by integrating along the interfacial curve the product of G and the strength of a differential source at $(R'(\phi', t'), t')$ on the interface. The expression for the temperature distribution obtained by this method can then be integrated with respect to t' between the limits $t' = 0$ to $t' = t$ to yield the temperature distribution in the system with the moving surface source liberating heat continuously for $t > 0$ along the interface.

To obtain the strength of a differential source at point $(R(\phi', t'), t')$ on the curve the following procedure was used: An infinitesimal length (dl') of the curve at $(R(\phi', t'), t')$ is equal to $\sqrt{[(\partial R/\partial \phi')^2 + R^2]}.d\phi'$. The normal velocity of the moving source at $(R . \phi')$ is

$$\left(\frac{\partial R}{\partial t'}\vec{i}_{r'}\right).\left(\vec{i}_{r'} - \frac{1}{R}\frac{\partial R}{\partial \phi'}\vec{i}_{\phi'}\right)\Big/\sqrt{\left[1 + \frac{1}{R^2}\left(\frac{\partial R}{\partial \phi'}\right)^2\right]}$$

or

$$R\frac{\partial R}{\partial t'}\Big/\sqrt{\left[R^2 + \left(\frac{\partial R}{\partial \phi'}\right)^2\right]}$$

where $\vec{i}_{r'}$ and $\vec{i}_{\phi}$ are unit normals in the r' and ϕ' directions with the direction of the normal velocity in the direction away from the solid phase.

The area covered by the infinitesimal length (dl') of the moving source during time dt' is $R(\partial R/\partial t')\, d\phi'\, dt'$. The latent heat liberated by the differential source during time dt' at (R, ϕ') on the interfacial curve is

$$L\rho R(\partial R/\partial t')\, d\phi'\, dt'.$$

The strength of the differential source is defined as

$$\frac{1}{\rho c(T_F - T_{wx})}.L\rho R\frac{\partial R}{\partial t'}\, d\phi'\, dt'$$

where the temperature difference $(T_F - T_{wx})$ is introduced in the preceding expression to normalize the temperature.

The Q-solution is therefore, given by

$$Q(r, \phi, t) = \int_0^t dt' \int_0^{\phi_0} \beta . R\frac{\partial R}{\partial t'} G(r, \phi', r', \phi'; t, t')\, d\phi'.$$

Substituting equation (31b) for G gives:

$$Q(r, \phi, t) = \frac{\beta}{\phi_0 \alpha}\sum_p \times \sin p\phi \int_0^t \frac{dt}{(t - t')}\int_0^{\phi_0} R . \frac{\partial R}{\partial t'} . \sin p\phi' \times \exp\left(-\frac{r^2 + R^2}{4\alpha(t - t')}\right) . I_p\left(\frac{rR}{2\alpha(t - t')}\right) d\phi'. \quad (36)$$

The expression for $Q(r, \phi, t)$ can be written in terms of the dimensionless variables $\eta = R/\sqrt{(4\alpha t')}$, r^*, and $\tau = t'/t$. The choice of these dimensionless variables is dictated by the similarity conditions in the problem. Equation (10) for the interface becomes in terms of η:

$$\eta = R'(\phi') \quad (37)$$

and the expression for Q becomes in terms of the dimensionless variables

$$Q(r^*, \phi) = \frac{2\beta}{\phi_0}\sum_p \times \sin p\phi \int_0^1 \frac{d\tau}{1 - \tau}\int_0^{\phi_0} \eta^2 \sin p\phi' \times \exp\left(-\frac{\eta^2\tau + r^{*2}}{1 - \tau}\right).I_p\left(\frac{2\eta(\sqrt{\tau}).r^*}{1 - \tau}\right) d\phi' \quad (38)$$

where η is related to ϕ' by equation (37).

Once the function $R'(\phi')$ is determined (or specified) the integrals in the Q-solution can be evaluation. numerically. For convenience in numerical computation equation (38) can be reordered and written as

$$Q(r^*, \phi) = \frac{2\beta}{\phi_0}\int_0^1 \frac{d\tau}{1 - \tau}\int_0^{\phi_0} \eta^2 \times \exp\left(-\frac{\eta^2\tau + r^{*2}}{1 - \tau} + \frac{2\eta(\sqrt{\tau})r^*}{1 - \tau}\right)$$

$$\times \left[\sum_{p} \sin p\phi \sin p\phi' \exp\left(-\frac{2\eta(\sqrt{\tau})r^*}{1-\tau} \right) \times I_p\left(\frac{2\eta(\sqrt{\tau})r^*}{1-\tau} \right) \right] . \mathrm{d}\phi'.$$

The infinite series in the above equation was truncated at that value of p at which

$$\exp\left(-\frac{2\eta(\sqrt{\tau})r^*}{1-\tau} \right) . I_p\left(\frac{2\eta(\sqrt{\tau})r^*}{1-\tau} \right) < 10^{-6}.$$

The double integral in the equation was evaluated by using the Gaussian–Legendre quadrature formula twice, once for τ and once for ϕ'. A twenty-point Gaussian quadrature formula was used for τ in the range $0 < \tau < 1$ and a forty-point quadrature formula was used for ϕ' in the range $0 < \phi' < \phi_0$. Increasing the number of abscissa points in the quadrature formula affects the value of the Q-solution only in the fifth decimal place.

The P- and the Q-solutions can now be combined according to equation (22) to give the dimensionless temperature distribution $T(r^*, \phi)$

$$T(r^*, \phi) = \frac{8T_i^*}{\phi_0} \int_0^\infty v e^{-(r^*-v)^2} \times \left[\sum_{p'} \frac{\sin p'\phi}{p'} e^{-2vr^*} I_{p'}(2vr^*) \right] \mathrm{d}v + T_{wx}^*\left(1 - \frac{\phi}{\phi_0}\right) + T_{wy}^* \frac{\phi}{\phi_0} - \frac{\sqrt{\pi}}{\phi_0} r^* \sum_p (T_{wx}^* - T_{wy}^* \cos p\phi_0) . \frac{\sin p\phi}{p} \times e^{-r^{*2}/2} . \left[I_{(p+1)/2}\left(\frac{r^{*2}}{2}\right) + I_{(p-1)/2}\left(\frac{r^{*2}}{2}\right) \right] + \frac{2\beta}{\phi_0} \sum_p \sin p\phi \int_0^1 \frac{\mathrm{d}\tau}{1-\tau} \int_0^{\phi_0} \eta^2 \sin p\phi' \times \exp\left(-\frac{\eta^2\tau + r^{*2}}{1-\tau} \right) . I_p\left(\frac{2\eta(\sqrt{\tau})r^*}{1-\tau} \right) \mathrm{d}\phi' \tag{39}$$

where

$$T = T_s^* \text{ for } r^* \leqslant \eta \qquad 0 < \phi < \phi_0$$
$$T = T_L^* \text{ for } r^* \geqslant \eta \qquad 0 < \phi < \phi_0.$$

The equation for the interface curve must be known before the above expression can be used to calculate the temperature distribution. To obtain an analytical solution for the interface curve, the condition at the interface given by equation (18) can be used. This condition gives:

$$\frac{8T_i^*}{\phi_0} \int_0^\infty v e^{-(r^*-v)^2} \times \left[\sum_{p'} \frac{\sin p'\phi}{p'} e^{-2vr^*} I_{p'}(2vr^*) \right] \mathrm{d}v + T_{wx}^*\left(1 - \frac{\phi}{\phi_0}\right) + T_{wy}^* \frac{\phi}{\phi_0} - \frac{\sqrt{\pi}}{\phi_0} r^* \sum_p (T_{wx}^* - T_{wy}^* \cos p\phi_0) \times \frac{\sin p\phi}{p} . e^{-r^{*2}/2} \left[I_{(p+1)/2}\left(\frac{r^{*2}}{2}\right) + I_{(p-1)/2}\left(\frac{r^{*2}}{2}\right) \right] + \frac{2\beta}{\phi_0} \sum_p \sin p\phi \int_0^1 \frac{\mathrm{d}\tau}{1-\tau} \int_0^{\phi_0} \eta^2 \sin p\phi' \exp\left(-\frac{\eta^2\tau + r^{*2}}{1-\tau} \right) . I_p\left(\frac{2\eta\sqrt{\tau r^*}}{1-\tau} \right) \mathrm{d}\phi' = 0. \tag{40}$$

The point designated by co-ordinates (r^*, ϕ) in equation (40) lies on the interface, i.e. for any given ϕ, $r^* = R'(\phi)$. Equation (40) is a non-linear integro-differential equation for η or $R'(\phi')$. In principle, equation (40) could be solved to get the expression for $R'(\phi')$, which in turn would give the equation for the interfacial curve. However, because of its complexity equation (40) is not amenable to an analytic solution. Instead, an equation for the interfacial curve was assumed initially and from it the expression for $R'(\phi')$ was obtained.

The equation for the interfacial curve may contain n parameters which can be evaluated by solving n-simultaneous equations, obtained by writing equation (40) for n points on the interface. Because this procedure is terribly cumbersome, a one-parameter hyperbola was used to represent the interface and the parameter was evaluated by trial and error to satisfy equation (40) at one point P on the interface as shown in Figs. 1 and 2. *A posteriori* it will be shown that

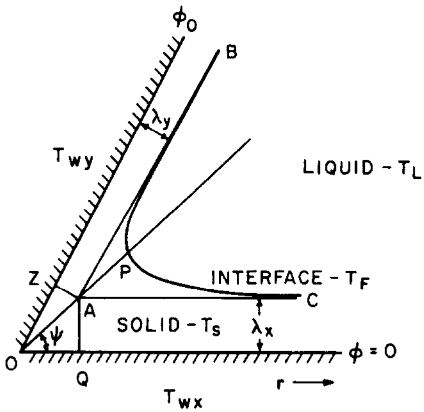

FIG. 1. Schematic sketch illustrating solidification of a liquid in a wedge ($\phi_0 < 180°$).

the one-parameter hyperbola obtained in this manner satisfies the condition imposed by equation (40) at other points on the interface.

From the physical condition of the problem it is apparent that the hyperbola chosen to represent the interface must be asymptotic to the lines AB and AC in Figs. 1 and 2. Far away from the corner of the wedge the surfaces $\phi = 0$ and $\phi = \phi_0$, respectively, will produce interfaces of AB and AC which can be predicted from the one-dimensional Neumann's solution [1]. The distances, λ_x or λ_y in Figs. 1 and 2, which characterize the positions of the solid–liquid interface for the one-dimensional case when the

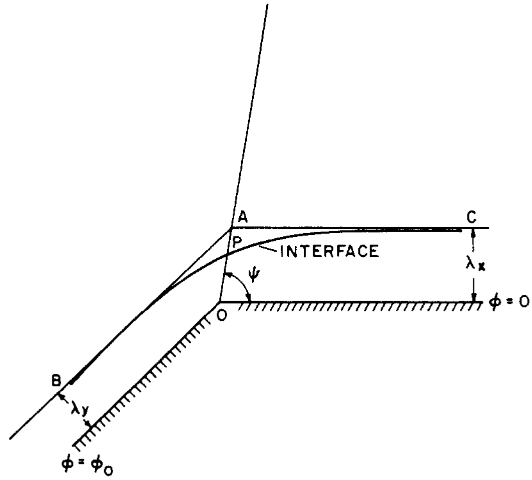

FIG. 2. Schematic sketch illustrating solidification of a liquid outside a wedge ($\phi_0 > 180°$) with unequal surface temperatures.

thermal diffusivities of the liquid and solid phases are equal, are given [1] by the equation:

$$-\frac{T_w e^{-\lambda^2}}{\operatorname{erf}\lambda} - \frac{T_i^* e^{-\lambda^2}}{\operatorname{erfc}\lambda} = (\sqrt{\pi})\beta\lambda \qquad (41)$$

where $\lambda \equiv \lambda_x$ for $T_w = T^*_{wx}$ and

$$\lambda \equiv \lambda_y \text{ for } T_w = T^*_{wy}.$$

Obviously, when the surfaces of the wedge are at equal temperatures, $\lambda_x = \lambda_y = \lambda$ and $T^*_{wx} = T^*_{wy} = T_w$.

The equation for the interface can be obtained by writing the equation for a hyperbola with A as its pole and the shifting the pole from A to the corner of the wedge. With λ_x and λ_y known, the equation for the one-parameter hyperbola interface is given by the equation:

$$(\eta\sin\phi' - \lambda_x)^2(\sin^2\phi_0/2 - \cos\phi_0/2)(\cot^1\phi_0/2) + 2(\eta\sin\phi' - \lambda_x)(\eta\cos\phi' - \lambda_x\cot\psi)\cot\phi_0/2 = a^2 \qquad (42)$$

where,

$$\psi \equiv \cot^{-1}\left(\frac{(\lambda_x/\lambda_y) + \cos\phi_0}{\sin\phi_0}\right)$$

and a is the flexible parameter in equation (42).

The function for η or $R'(\phi')$ is now given in implicit form by equation (42) but, as mentioned previously, the characteristic parameter a must be evaluated by trial and error.

To determine the value of the parameter a, initially it was taken as zero corresponding to the two asymptotic solutions intersecting. Then, a was incremented in steps of 0·2 until the left hand side of equation (40) was positive and negative for two consecutive values of a. Finally, linear interpolation was used to obtain the next trial value for a. This process was repeated until the left-hand side of equation (40) was zero to within a tolerance of $\pm 0{\cdot}001$. With a specified, the interfacial curve can be plotted by using equation (42) and a check whether equation (40) is satisfied at points other than P on the interfacial curve can be made.

For all cases discussed in the last section, the value of a which made the left hand side of equation (40) equal to zero at point P within the tolerance specified above, was within 10 per cent of the value required to satisfy the interfacial conditions at any other point r^*, ϕ. Thus, as shown in more detail in Sec. III, for all practical purposes a one-parameter hyperbola can represent the interfacial curve with satisfactory accuracy for most engineering calculations.

Once the interfacial curve is known, the temperature distribution in the liquid and solid phases are given by:

$$T_s = (T_F - T_{wx})T(r^*, \phi) + T_F \quad \text{for } r^* \leqslant R'(\phi), 0 < \phi < \phi_0 \qquad (43)$$

and

$$T_L = \frac{k_L}{k_s}(T_F - T_{wx})T(r^*, \phi) + T_F \quad \text{for } r^* \geqslant R'(\phi), 0 < \phi < \phi_0, \qquad (44)$$

where (r^*, ϕ) is given by equation (39).

Simplification of Q*-solution when the temperature of the two wedge surfaces are equal*

When the face temperatures are equal, the Q-solution given by equation (38) can be simplified by using the symmetry characteristics of the solutions about the line $\phi = \phi_0/2$. From equations (37) and (42) the symmetry property gives for the interface curve:

$$\eta = R'(\phi') = R'(\phi_0 - \phi'); \frac{\phi_0}{2} \leqslant \phi' < \phi_0. \qquad (45)$$

To make use of the symmetry condition about the line $\phi = \phi_0/2$, the integral in equation (38) is separated into two parts, or

$$Q = \frac{2\beta}{\phi_0}\sum_p \sin p\phi \int_0^1 \frac{d\tau}{1-\tau} \left[\int_{\phi_0/2}^{\phi_0} \eta^2 \sin p\phi' \exp\left(\frac{\eta^2(\sqrt{\tau}) + r^{*2}}{1-\tau}\right) \times I_p\left(\frac{2\eta(\sqrt{\tau})r^*}{1-\tau}\right) d\phi' + \int_0^{\phi_0/2} \eta^2 \sin p\phi' \exp\left(\frac{\eta^2(\sqrt{\tau}) + r^{*2}}{1-\tau}\right) \times I_p\left(\frac{2\eta(\sqrt{\tau})r^*}{1-\tau}\right) d\phi'\right]. \qquad (46)$$

The first integral with respect of ϕ' on the right-hand side of equation (46) can be simplified further by defining a new variable $\Theta = \phi_0 - \phi'$. In terms of this new variable the first integral becomes

$$-\int_{\phi_0/2}^{0} \eta^2 \sin p(\phi_0 - \Theta)\exp\left(\frac{-\eta^2\sqrt{\tau} + r^{*2}}{1-\tau}\right) \times I_p\left(\frac{2\eta(\sqrt{\tau})r^*}{1-\tau}\right) d\Theta \qquad (47)$$

and equation (45) becomes in terms of Θ

$$\eta = R'(\Theta).$$

Changing the integration variable from Θ to ϕ' in (47), then substituting for the first integral in equation (46), the expression for Q reduces to

$$Q = \frac{2\beta}{\phi_0}\sum_p \sin p\phi \int_0^1 \frac{d\tau}{1-\tau} \times \int_0^{\phi_0/2} \eta^2 \exp\left(-\frac{\eta^2\tau + r^{*2}}{1-\tau}\right) \times I_p\left(\frac{2\eta(\sqrt{\tau})r^*}{1-\tau}\right)[\sin p(\phi_0 - \phi') + \sin p\phi']\, d\phi'.$$

For n odd, $\sin p(\phi - \phi') + \sin p\phi' = 2 \sin p\phi'$ and for n even, $\sin p(\phi - \phi') + \sin p\phi' = 0$.

Therefore, the expression for Q simplifies to

$$Q = \frac{4\beta}{\phi_0}\sum_{p'} \sin p'\phi \int_0^1 \frac{d\tau}{1-\tau} \int_0^{\phi_0/2} \eta^2 \exp\left(-\frac{\eta^2\tau + r^{*2}}{1-t}\right) \times I_p'\left(\frac{2\eta(\sqrt{\tau})r^*}{1-\tau}\right). \sin p'\phi'\, d\phi'. \quad (48)$$

3. RESULTS AND DISCUSSION

As shown in the preceding section, the solution to the problem of freezing or melting in a wedge-shaped corner section includes the location of the interface and the temperature distribution in the solid and liquid phases. Using a single parameter hyperbola with a characteristic parameter a, once the value of this parameter has been determined from the appropriate integro-differential equation, the interfacial curve can be represented by equation (42) and the temperature distributions in the solid and liquid phases are given by equations (43) and (44), respectively.

The solution contains the parameters ϕ_0, β, T_i^*, T_{wy}^*, r^* and ϕ and their ranges are:

$$0 < \phi_0 < 2\pi$$
$$\beta \geqslant 0$$
$$T_i^* \geqslant 0$$
$$T_{wy}^* \leqslant 0$$
$$0 < r^* < \infty \qquad 0 < \phi < \phi_0.$$

It is difficult to present results for all possible combinations of independent parameters in a comprehensive form. However, the method presented here can be used to make calculations for any combination and their accuracy can be estimated from the results of several examples which can be compared with data in the literature.

In the presentation below the simplest case is considered first and subsequent examples follow in an ascending order of difficulty.

Example 1

The liquid in an internal corner (90° wedge) is at fusion temperature with the surfaces of the wedge maintained at *equal* temperature lower than the fusion temperature. The value of the pertinent parameters are: $\phi_0 = 90°$, $\beta = 1{\cdot}5613$, $T_i^* = 0$, $T_{wx}^* = T_{wy}^* = -1$.

The location of the interface is shown in Fig. 3.

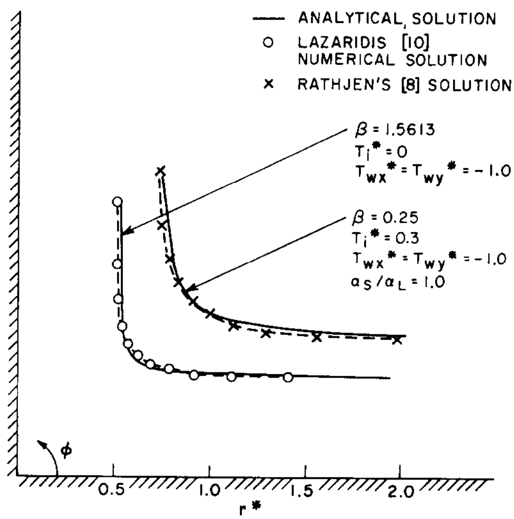

FIG. 3. Interfacial curve for internal corner with the surfaces maintained at equal temperatures: (a) Liquid initially at the fusion temperature; (b) Liquid initially at a uniform temperature above the melting point. (Comparison of analysis with results from [8] and [10].)

For comparison the numerical results of Lazaridis [10] for this case are also plotted in the same figure. The interfacial curve obtained from the general analytical solution agrees with Lazaridis' numerical solution to within 5 per cent.

It should be noted that when the liquid is initially at the fusion temperature there is no diffusion of heat in the liquid phase and the solution is, therefore, independent of the ratio α_s/α_L. In such a case, the analytical solution presented in this paper is exact for any value of α_s/α_L.

Example 2

Solidification of a liquid initially at a temperature *higher than the fusion temperature* with the surfaces of the wedge at *equal* temperature but lower than the fusion temperature.

For such cases the analytical solution is exact when the ratio of the thermal diffusivity in the solid to that in the liquid phase is unity. For the cases where the ratio α_s/α_L is not unity, the analytical solution is not exact but will, as shown later, give a good approximation for the location of the interface with some empirical corrections.

With the ratio of diffusivities is equal to unity and $\phi_0 = 90°$, $T_i^* = 0{\cdot}3$, $\beta = 0{\cdot}25$, the location of the interface and the plot of T at $\phi = 45°$ vs. r^* are presented in Figs. 3 and 4, respectively. The results from the general analytical solution agree with those obtained by Rathjen [8] to within 5 per cent.

In Figs. 5 and 6, the interfacial curves for T_i^* and several β's are plotted for wedge angles 60 and 270 degrees respectively. No numerical or experimental results for these wedge angles are available in the literature for comparison.

When the ratio of thermal diffusivities is not equal to unity, a first approximation for the location of the interface can be obtained from the analytical solution. An inprovement in this approximation can be made by using the following empirical approach.

It is known that either for equal or for unequal thermal diffusivities of the two phases, the interfacial curve at distances far away from the corner must be asymptotic to the lines representing the interfaces for the one dimensional cases. For the analytical solution the interfacial curve for the wedge has been made

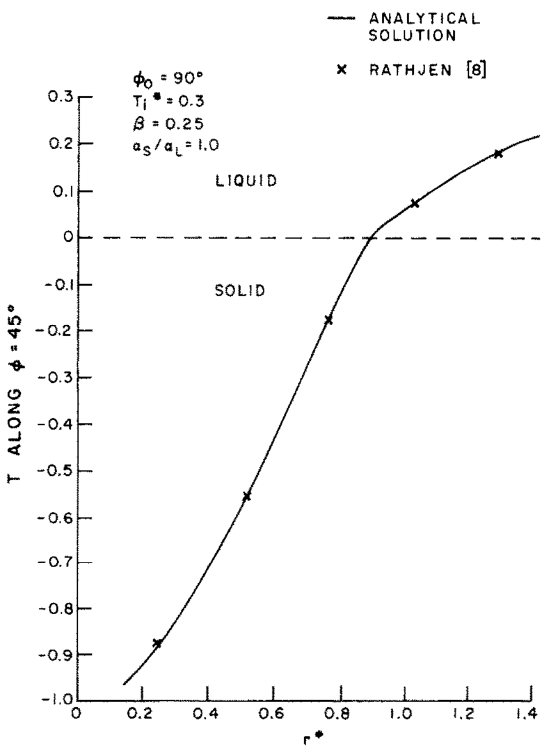

FIG. 4. Temperature along $\phi = 45°$ vs. r^* for internal corner.

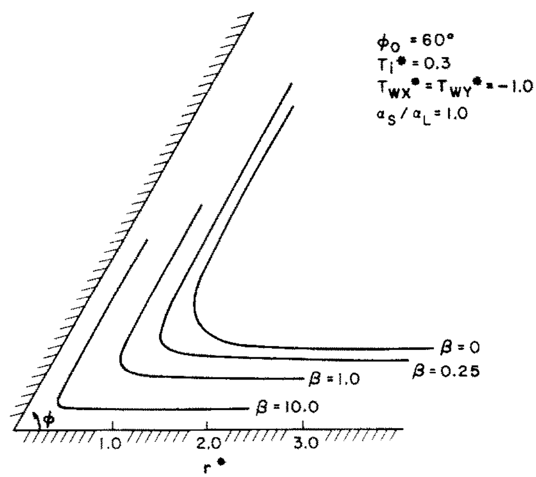

FIG. 5. Interfacial curves for solidification in a 60°-wedge.

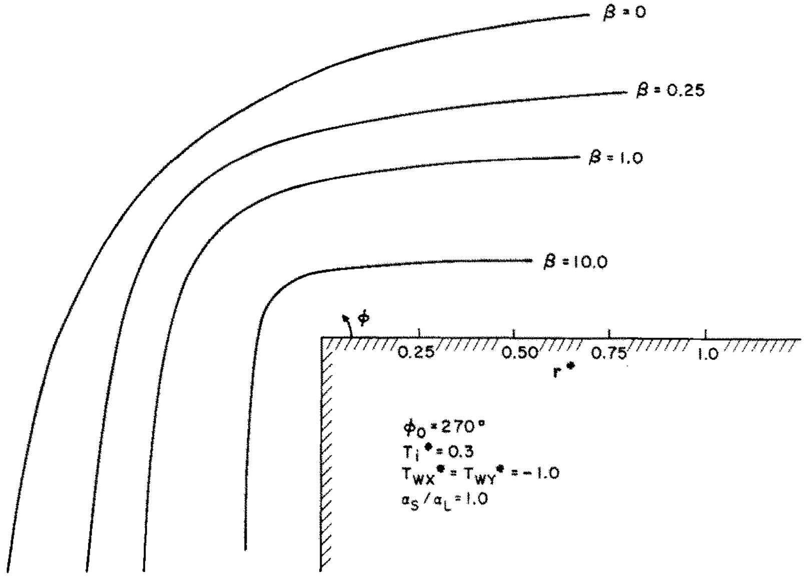

FIG. 5. Interfacial curves for solidification outside a 270°-wedge.

asymptotic to the lines representing the one-dimensional interfaces for *equal* thermal diffusivities. A reasonable approximation of the interfacial curve for unequal thermal diffusivities can be obtained by shifting the analytical solution curve in such a manner that the analytical curve becomes asymptotic to the lines representing the one-dimensional interfaces for *unequal* thermal diffusivities. This shift can be made by calculating the radial coordinate η' of the new curve which approximates the interfacial curve for unequal thermal diffusivities, from the radial coordinate η of the analytical solution curve using the equations

$$\eta' = \eta - \frac{\lambda_x - \lambda_x'}{\sin\phi'}; \quad 0 < \phi' \leqslant \phi_0/2 \tag{49a}$$

$$\eta' = \eta - \frac{\lambda_y - \lambda_y'}{\sin(\phi_0 - \phi')}; \quad \phi_0/2 \leqslant \phi' < \phi_0 \tag{49b}$$

where λ_x' and λ_y' are the characteristic distances for the line representing the one-dimensional interfaces for unequal thermal diffusivities and are obtained from the equation [1]

$$-\frac{T_w\,e^{-\lambda'^2}}{\operatorname{erf}\lambda'} - \frac{T_i^*(\alpha_s/\alpha_L)^{\frac{1}{2}}\,e^{-(\alpha_s/\alpha_L)\lambda'^2}}{\operatorname{erfc}[(\alpha_s/\alpha_L)^{\frac{1}{2}}\lambda']} = (\sqrt{\pi})\beta\lambda' \tag{50}$$

where $\lambda \equiv \lambda_x'$ for $T_w = T_{wx}^*$ and $\lambda \equiv \lambda_y'$ for $T_w = T_{wy}^*$. When $\alpha_s/\alpha_L = 1$, equation (50) reduces to equation (41).

When the shift is made according to equation (50), two values for η' at $\phi' = \phi_0/2$ are obtained for the case of unequal temperatures at the surfaces of the wedge. An arithmetic average of these two values can be taken to get a unique value for η' at $\phi' = \phi_0/2$. However, for equal temperatures at the wedge surfaces only one value for η' at $\phi' = \phi_0/2$ will be obtained from the equations.

For example, for the case of the temperatures at the surfaces of a 90 degree wedge equal, with $\beta = 0{\cdot}6$, $T_i^* = 0{\cdot}5$, and $\alpha_s/\alpha_L = 2{\cdot}5$, the interfacial curve obtained by Rathjen [8] using a numerical method is plotted in Fig. 7. The curves obtained by the original analytical solution and after making the shift according to equation (50) are plotted in the same figure.

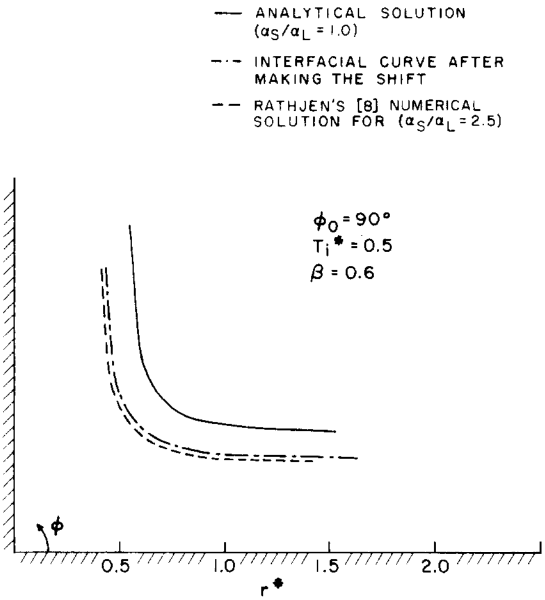

FIG. 7. Interfacial curves for solidification of a material with unequal thermal diffusivities in an internal corner.

Although the curve from the original analytical solution deviates from Rathjen's results by as much as 30 per cent, the curve obtained after making the shift deviates from that of Rathjen's by no more than 5 per cent.

The results from the general analytical solution with the appropriate shift of the interfacial curve to correct for unequal thermal diffusivities predicts the experimental measurements of Jiji *et al* [9] for freezing of water ($\alpha_s/\alpha_L = 9{\cdot}2$) in an internal corner to within 5 per cent. The experimental and the calculated values of x_0 (see Fig. 8 for definition of x_0) at various times for three pairs of T_i^* and β are plotted as x_0 vs. $\sqrt{t}$ in Fig. 8. The curves are straight lines because of the similarity conditions.

Example 3

A liquid is initially at a temperature higher than its fusion temperature, but the surfaces of the wedge are maintained at *unequal* temperatures lower than the fusion temperature.

The value of the pertinent parameters for this case are: $\phi_0 = 90°$, $\beta = 0{\cdot}5$, $T_i^* = 0{\cdot}2751$, $T_i^* = -0{\cdot}7143$ and $\alpha_s/\alpha_L = 1{\cdot}111$.

The interfacial curve obtained from the analytical solution is plotted in Fig. 9. It agrees with Lazaridis' [10] numerical solution to within 5 per cent. If the necessary shift in the interfacial curve is made to account for unequal thermal diffusivities, then the agreement with Lazaridis' numerical solution is even better than 5 per cent.

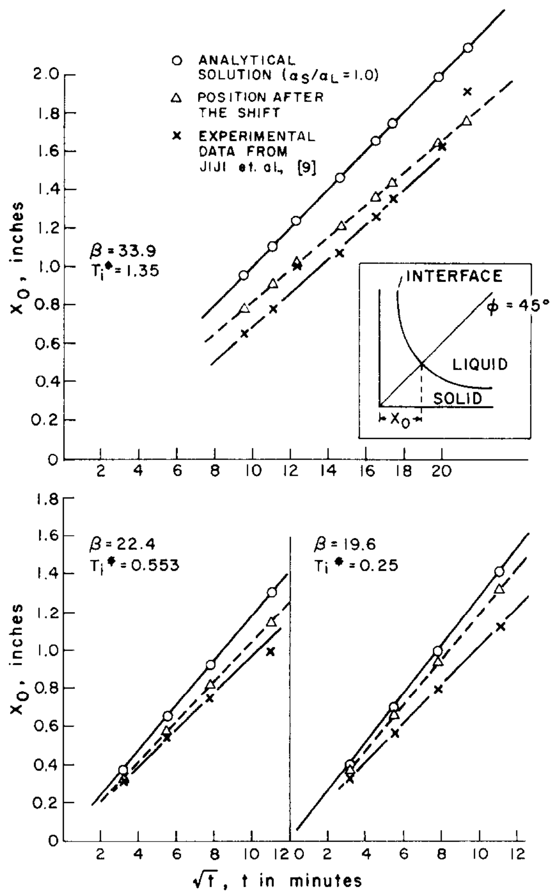

FIG. 8. Plot of location of interfacial curve along the diagonal vs. $\sqrt{t}$ for solidification of water in a square container.

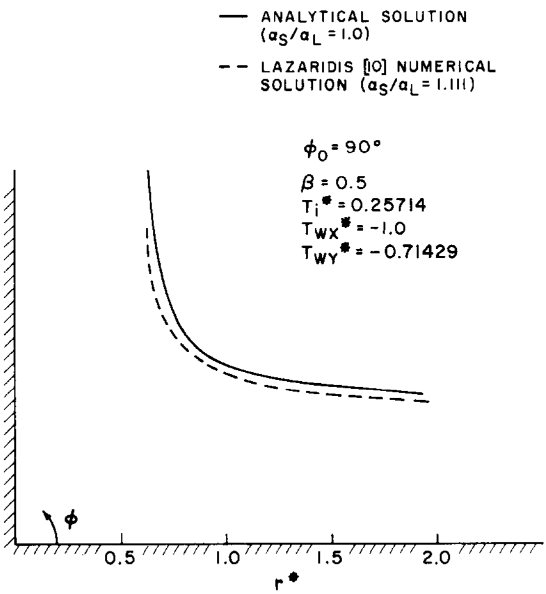

FIG. 9. Interfacial curves for internal corner with the liquid initially at uniform temperature and the surfaces maintained at unequal temperatures.

4. CONCLUSIONS

The process of freezing and melting in a wedge shaped enclosure has been analyzed for the conditions that the initial temperature of the freezing liquid or the melting solid is uniform and the wedge surfaces are maintained at uniform, but not necessarily equal temperatures. The results of this analysis agree within 5 per cent with results of numerical and experimental investigations. The analysis presented in this paper yields equations for the shape of the solid–liquid interface and the temperature distribution in the solid and in the liquid phase, assuming that conduction is the dominant mode of heat transfer in both phases. To extend the range of applicability of the analytical results presented here, it is recommended that the influence of convection in the liquid on the process be investigated.

ACKNOWLEDGEMENTS

The authors would like to express their appreciation to Professors R. D. Richtmyer and W. D. Fults of the Department of Mathematics and Professor J. F. Carley of the Department of Chemical Engineering at the University of Colorado for helpful suggestions during this study. They also wish to thank the Monsanto Company, the Gulf Oil Corporation and E. I. duPont de Nemours and Co, Inc. for financial support.

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TRANSFERT THERMIQUE AVEC FUSION OU SOLIDIFICATION DANS UN DIEDRE

Résumé— Le but de cette recherche était d'obtenir une solution analytique de la distribution de température et du mouvement de l'interface dans un liquide pur ou un alliage eutectique qui se solidifie ou fond dans un espace en forme de dièdre. Dans cette analyse, on suppose que la température initiale du milieu est uniforme et que les surfaces du dièdre sont maintenues à des températures uniformes mais non nécessairement égales.

La solution du problème a été obtenue par la superposition des solutions de deux problèmes auxiliaires. Le premier est le problème de la conduction thermique sans changement de phase, mais avec les mêmes conditions initiales et aux limites que celles du problème étudié. Le second sous-problème est celui de la conduction thermique avec changement de phase, mais pour des températures initiales et aux limites égales à zéro. Dans ce dernier sous-problème, la chaleur latente libérée due au changement de phase est représentée par une source surfaçique mobile le long de l'interface. Les distributions de température pour ces problèmes auxiliaires ont été obtenues à l'aide de la fonction de Green. La solution analytique présentée ici s'accorde à mieux que 5 pour cent avec des résultats expérimentaux ou numériques déjà publiés.

WÄRMEÜBERTRAGUNG MIT SCHMELZEN ODER ERSTARREN IN EINEM KEILFÖRMIGEN HOHLRAUM

Zusammenfassung— Das Ziel dieser Untersuchung war eine analytische Lösung für die Temperaturverteilung und die Bewegung der Trennfläche in einer reinen Flüssigkeit oder einer eutektischen Legierung, die in einem keilförmigen Hohlraum schmilzt oder sich verfestigt. In der Analyse wurde angenommen, dass die Anfangstemperatur des Mediums gleichförmig ist und dass die Flächen des Keils auf gleichförmigen, aber nicht notwendigerweise auf gleichen Temperaturen gehalten werden.

Die Lösung für dieses Problem erhält man, indem man die Lösungen für zwei Hilfsprobleme überlagert. Das erste war das Problem der Wärmeleitung ohne Phasenänderung aber mit denselben Anfangs- und Randbedingungen wie im vorliegen Fall. Das zweite war Wärmeleitung mit Phasenänderung, aber mit den Anfangs- und Randtemperaturen gleich Null, wobei Wärme frei gesetzt wird entsprechend einer Phasenänderung, die durch eine bewegte Trennfläche mit gleichförmiger Quellenverteilung dargestellt wird. Für die Lösung dieser Hilfsprobleme wurde die Green'sche Funktion benutzt. Die Ergebnisse dieser analytischen Lösung stimmen mit früheren Veröffentlichungen über experimentelle und numerische Ergebnisse für Spezialfälle bis auf 5 Prozent genau überein.

ТЕПЛООБМЕН ПРИ ПЛАВЛЕНИИ И ОТВЕРДЕВАНИИ В КЛИНООБРАЗНОЙ ФОРМЕ

Аннотация—Целью данного исследования был аналитический расчет распределения температур и движения границь раздела фаз в чистой жидкости или эвтектическом сплаве при плавлении или затвердевании в клинообразной форме. В процессе анализа предполагалось, что начальная температура среды и температура поверхности клина постоянны, но необязательно равны. Задача решалась путем наложения решений двух вспомагательных задач: задачи теплопроводности при отсутствии фазового перехода при тех же начальных и граничных условиях и задачи теплопроводности при фазовых изменениях с нулевыми начальными и граничными условиями. В последенем случае скрытая теплота, выделенная при фазовом изменении, представлена поверхностным источником, движущимся вдоль поверхности раздела. Распределения температуры для этих вспомогательных задач получены с помощью функции Грина. Представленные здесь результаты аналитического решения согласуются с ранее опубликованными экспериментальными и численными результатами для отдельных, случаев с точностью до 5%.